Rips Magnitude

Dejan Govc (joint with Richard Hepworth)
dejan.govc@abdn.ac.uk

1495 UNIVERSITY OF ABERDEEN

Magnitude

Definition (Leinster, 2008). *(X,d) finite metric space. A* weighting *of X is $w : X \to \mathbb{R}$ such that*

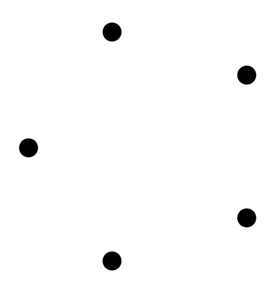
$$\sum_{x \in X} e^{-d(x,y)} w(x) = 1.$$

Define magnitude *and* magnitude function *of X:*

$$|X| = \sum_{x \in X} w(x), \qquad \mathrm{Mag}_X(t) = |tX|.$$

Example: Euclidean Cycle EC_n

Let EC_n consist of n equidistant points in S^1.

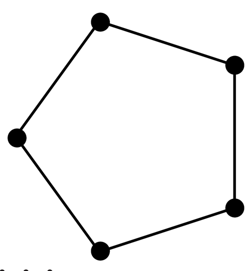


$$\mathrm{Mag}_{EC_5}(t) = \frac{5}{1 + 2e^{-\xi_1 t} + 2e^{-\xi_2 t}}$$

$$\xi_{1,2} = \sqrt{\frac{1}{2}(5 \pm \sqrt{5})}$$

Example: Cycle Graph C_n

Let C_n be the n-cycle graph with graph metric. Let $q = e^{-t}$.



$$\mathrm{Mag}_{C_5}(t) = \frac{5}{1 + 2e^{-t} + 2e^{-2t}} = 5 - 10q + 10q^2 - 20q^4 + \dots$$

Magnitude of a Graph

Theorem (Leinster, 2014). *The magnitude of a graph G is an integer power series in $q = e^{-t}$:*

$$\mathrm{Mag}_G(t) = \sum_{l=0}^{\infty} c_l q^l, \qquad c_l \in \mathbb{Z}.$$

Assuming $\forall i : x_{i-1} \neq x_i$, write $\mathbf{x} = (x_0, \dots, x_k)$ and $\ell(\mathbf{x}) = \sum_{i=1}^{k} d(x_{i-1}, x_i)$. Then, c_l are given by:

$$c_l = \sum_{k=0}^{\infty} (-1)^k \left| \{ \mathbf{x} \mid \ell(\mathbf{x}) = l \} \right|.$$

Magnitude Homology

Definition (Hepworth, Willerton, 2015). *The* magnitude chain complex *of G:*

$$\mathrm{MC}_{k,l}(G) = \langle \mathbf{x} \mid \ell(\mathbf{x}) = l \rangle$$

with boundary $\partial_{k,l} : \mathrm{MC}_{k,l}(G) \to \mathrm{MC}_{k-1,l}(G)$

$$\partial_{k,l}(\mathbf{x}) = \sum_{i=0}^{k} (-1)^i \partial_{k,l}^i(\mathbf{x}).$$

The homology of $(\mathrm{MC}_{*,l}, \partial_{*,l})$ is called the magnitude homology $\mathrm{MH}_{k,l}(G)$ *of G.*

Proposition (Hepworth, Willerton, 2015).

$$\mathrm{Mag}_G(t) = \sum_{l=0}^{\infty} \chi(\mathrm{MH}_{*,l}(G)) q^l \qquad (q = e^{-t})$$

Example: 5-Cycle Graph C_5

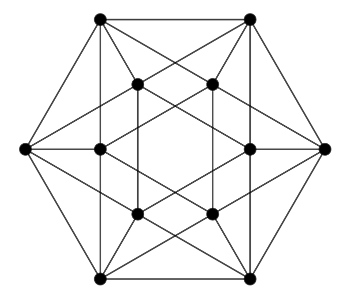
$\mathrm{MC}_{k,l}$	0	1	2	3	4
0	5				
1		10			
2		10	20		
3			40	40	
4			20	120	80

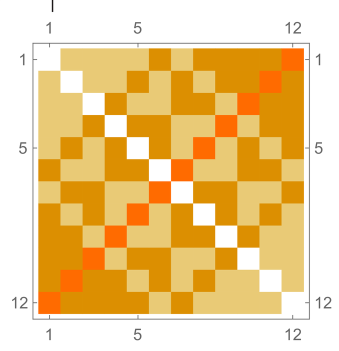
$\mathrm{MH}_{k,l}$	0	1	2	3	4
0	5				
1		10			
2			10		
3			10	10	
4				30	10

Example: Icosahedral Graph

$\mathrm{MC}_{k,l}$	0	1	2	3	4
0	12				
1		60			
2		60	300		
3		12	600	1500	
4			420	4500	7500

$\mathrm{MH}_{k,l}$	0	1	2	3	4
0	12				
1		60			
2			240		
3				912	
4					3420





Compact Metric Spaces

Definition (Meckes, 2010). *(X,d) compact positive definite metric space. The* magnitude *of X is:*

$$|X| = \sup\{|W| \mid W \subseteq X, W \text{ finite.}\}$$

Example (Leinster, Willerton, 2009).

- $|t \cdot (S^1, \textit{euclidean})| = \pi t + O(t^{-1})$ *as* $t \to \infty$
- $|t \cdot (S^1, \textit{geodesic})| = \frac{\pi t}{1 - e^{-\pi t}}$

Persistent Homology

Definition (various authors). *Given a* filtration functor *$S : (I, \leq) \to$ **SCx** $(I \overset{\text{loc.fin.}}{\subseteq} \mathbb{R})$, we call $F = H_k \circ S$ the k-th* persistent homology *of S.*

Definition. *Given (X,d) and $A \subseteq X$ finite, define* Rips filtration*: $A \in \mathcal{R}_r(X) \iff \operatorname{diam} A \leq r$.*

Rips Magnitude

Definition. *The* Rips magnitude *of (X,d) is the filtered Euler characteristic of $\mathcal{R}_r(X)$:*

$$\begin{aligned}\mathrm{RMag}_X(t) &= \sum_{\emptyset \neq A \subseteq X} (-1)^{|A|-1} e^{-t \operatorname{diam}(A)} \\ &= \sum_{i=0}^{n} \chi(\mathcal{R}_{r_i}(X))(e^{-r_i t} - e^{-r_{i+1} t}) \\ &= \sum_{j=1}^{m} (-1)^{|\beta_i|}(e^{-a_j t} - e^{-b_j t})\end{aligned}$$

Remark. *This generalizes beyond Rips filtrations.*

Rips Filtrations of Cycles

Theorem (Adamaszek, 2011). *Let $0 \leq r < \frac{n}{2}$ and let $\mathcal{R}_r(C_n)$ be the r-th stage of the Rips filtration corresponding to the n-cycle graph. Then*

$$\mathcal{R}_r(C_n) \simeq \begin{cases} \bigvee_{n-2r-1} S^{2l}; & r = \frac{l}{2l+1} n, \\ S^{2l+1}; & \frac{l}{2l+1} n < r < \frac{l+1}{2l+3} n. \end{cases}$$

Rips Magnitude of Cycles

Theorem. *Let $q = e^{-t}$. The Rips magnitude of an n-cycle graph can be expressed explicitly as:*

$$\mathrm{RMag}_{C_n}(t) = \sum_{\substack{\text{odd } r \mid n \\ r \neq n}} \frac{n}{r} q^{\frac{n}{r} \frac{r-1}{2}} (1 - q) + q^{\lfloor \frac{n}{2} \rfloor}.$$

Rips magnitude of EC_n (δ_r, $\delta_{r,n}$ arise as distances):

$$\mathrm{RMag}_{EC_n}(t) = \sum_{\substack{\text{odd } r \mid n \\ r \neq n}} \frac{n}{r} (e^{-\delta_r t} - e^{-\delta_{r,n} t}) + e^{-\delta_n t}.$$

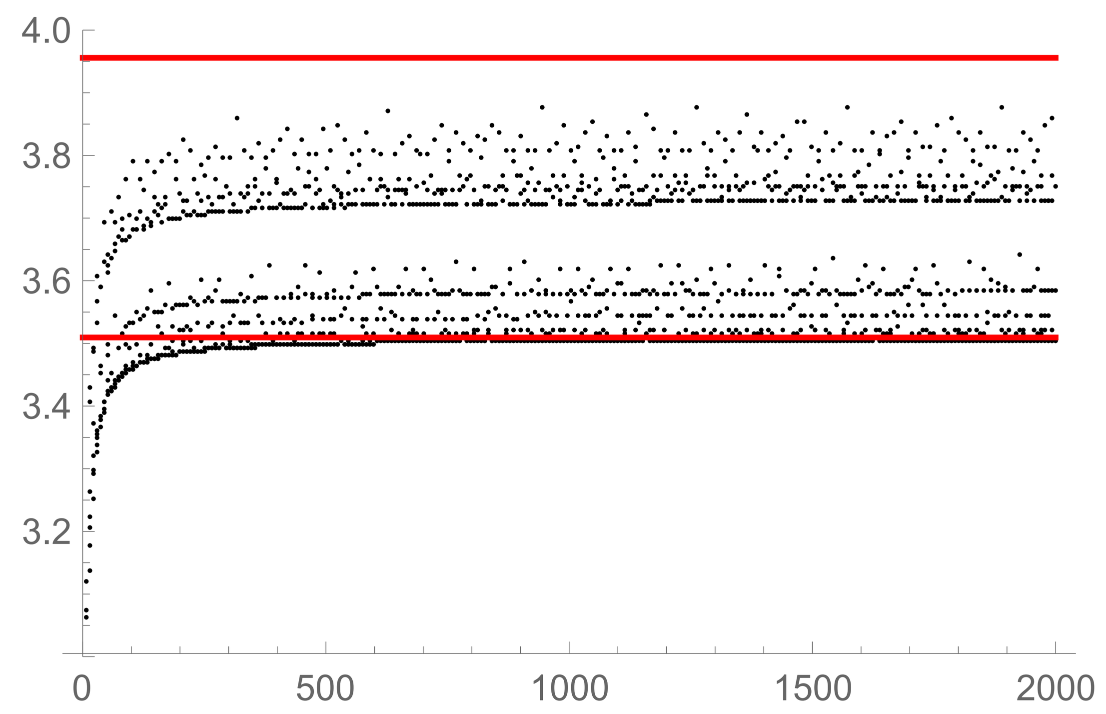
Euclidean Circle

Theorem. *The Rips magnitudes of EC_n satisfy*

$$\liminf_{n \to \infty} \mathrm{RMag}_{EC_n}(t) = e^{-2t} + 2\pi t,$$

$$\limsup_{n \to \infty} \mathrm{RMag}_{EC_n}(t) = e^{-2t} + 2\pi t \sum_{r \text{ odd}} \frac{1}{r} e^{-2t \cos \frac{\pi}{2r}} \sin \frac{\pi}{2r}.$$

For example, $\mathrm{RMag}_{EC_n}(\frac{1}{2})$ looks as follows:



Geodesic Circle

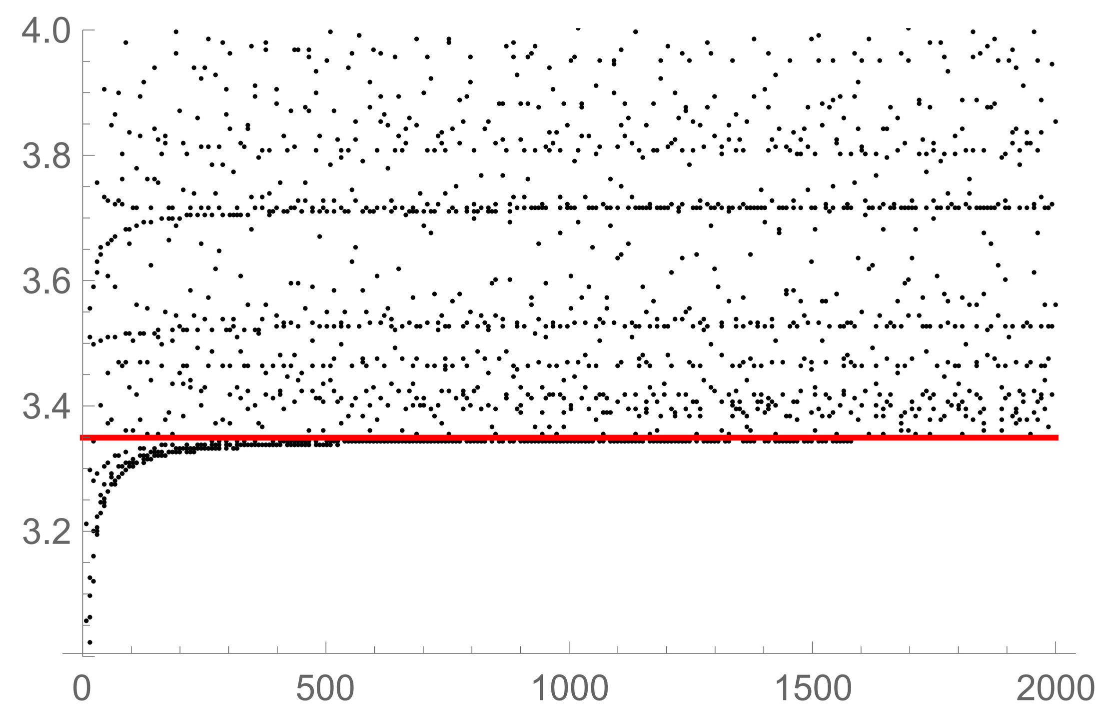
Normalize the cycle graph C_n so that the length is 2π. Call the resulting metric space GC_n.

Theorem. *The Rips magnitudes of GC_n satisfy*

$$\liminf_{n \to \infty} \mathrm{RMag}_{GC_n}(t) = e^{-\pi t} + 2\pi t,$$

$$\limsup_{n \to \infty} \mathrm{RMag}_{GC_n}(t) = \infty.$$

For example, $\mathrm{RMag}_{GC_n}(\frac{1}{2})$ looks as follows:



References

[1] T. Leinster & M. Meckes: *The Magnitude of a Metric Space: From Category Theory to Geometric Measure Theory*, Measure Theory in Non-Smooth Spaces (2017), 156–193.

[2] M. Adamaszek: *Clique Complexes and Graph Powers*, Israel Journal of Mathematics, **196**, No. 1 (2013), 295–319.

[3] R. Hepworth & S. Willerton: *Categorifying the Magnitude of a Graph*, Homology, Homotopy and Applications, **19**, No. 2 (2017), 31–60.

[4] T. Leinster & M. Shulman: *Magnitude Homology of Enriched Categories and Metric Spaces*, arXiv preprint, arXiv:1711.00802.

Funding

This work was funded by the EPSRC grant EP/P025072/1 (Topological Analysis of Neural Systems).